

Symmetric categorical grammar

Tuesday

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1. Computational Interpretation

Wadler (2003):

The computational interpretation of a sequent is as follows:

one must supply a value for every variable (and term) in the antecedent, and the computation will pass a value to some continuation variable (or coterminant) in the succedent.

2. Curry-Howard Correspondence

Lambek Calculi. CH correspondence: $\mathsf{T}(A \setminus B) = \mathsf{T}(B / A) = \mathsf{T}(A) \rightarrow \mathsf{T}(B)$.

$(\setminus R)$ and $(/R)$ correspond to abstraction (the $(\setminus I)$ and $(/I)$ of Natural Deduction):

$$\frac{\Gamma \circ x : A \xrightarrow{M} B}{\Gamma \xrightarrow{\lambda x.M} B/A} (/R) \quad \frac{x : A \circ \Gamma \xrightarrow{M} B}{\Gamma \xrightarrow{\lambda x.M} A \setminus B} (\setminus R)$$

$(\setminus L)$ and $(/L)$ correspond to function application (the $(\setminus E)$ and $(/E)$ of Natural Deduction).

$$\frac{\Delta \xrightarrow{N} B \quad \Gamma[y : A] \xrightarrow{M} C}{\Gamma[x : A/B \circ \Delta] \xrightarrow{M[y := x N]} C} (/L)$$

Note, in $(/R)$ (and $(\setminus R)$) we build the term of the active formula A/B ($B \setminus A$), whereas in $(/L)$ (and $(\setminus L)$) we substitute y by $x N$ (i.e., there is a hidden cut).

2.1. CH: Example

$$\frac{\frac{np : x_3 \vdash np : \mathbf{x}_3 \quad np : x_1 \circ np \backslash s : \mathbf{y} \vdash s : u[u := \mathbf{y} \mathbf{x}_1]}{np : x_1 \vdash np : \mathbf{x}_1 \quad s : u \vdash s : u} (\backslash L)}{np : x_1 \circ ((np \backslash s) / np : \mathbf{x}_2 \circ np : x_3) \vdash s : (y \ x_1)[y := \mathbf{x}_2 \ \mathbf{x}_3]} (/L)$$

$$(x_2 \ x_3) \ x_1$$

word	syntax	type	term	meaning
“sara”	np	e	s	sara
“left”	$np \backslash s$	$e \rightarrow t$	$\lambda x_e.(\text{left } x)_t$	$\{\text{sara}\}$
“teases”	$(np \backslash s) / np$	$e \rightarrow (e \rightarrow t)$	$\lambda x_e.(\lambda y_e.((\text{teases } x) \ y)_t)$	$\{\langle \text{sara}, \text{ilaria} \rangle\}$
“every boy”	$s / (np \backslash s)$	$(e \rightarrow t) \rightarrow t$	$\lambda y. \forall \lambda x. ((\text{boy } x) \Rightarrow (y \ x))$	$\{y \mid \llbracket \text{boy} \rrbracket \subseteq y\}$

Eg. “Sara teases Ilaria”: We replace the proof variables with the meaning representations of the words: x_1, x_2, x_3 by $s, \lambda x. \lambda y. ((\text{teases } x) \ y), i$, by β -red.:

$$((\text{teases } i) \ s)$$

2.2. Curry-Howard Correspondence for LG?

- Curien and Herbelin give a Curry-Howard Correspondence for Classical Logic with $\bar{\lambda}\mu\tilde{\mu}$ -calculus.
- Lambek-Grishin Calculus:
 - Restriction due to resource sensitivity: Linearity condition, the (co)variable bound in a (co)abstraction occurs free exactly once in the body.
 - Refinement: taking directionality into account. E.g., instead of only $\lambda x.M$, $x \backslash M$ and M/x (similarly for co-lambda).
- Wadler (2003): Terms yield values vs. Coterms consume values.

2.3. $\bar{\lambda}\mu\tilde{\mu}$ -terms

Terms (values) Co-terms (contexts) commands (cuts).

	$x \in \text{Term}^A$	if $x \in \text{Var}^A$
(L ABSTRACTION)	$x \backslash M \in \text{Term}^{B \setminus A}$	if $x \in \text{Var}^B, M \in \text{Term}^A$
(R ABSTRACTION)	$M / x \in \text{Term}^{A/B}$	if $x \in \text{Var}^B, M \in \text{Term}^A$
(L COAPPLICATION)	$K \prec M \in \text{Term}^{B \otimes A}$	if $K \in \text{CoTerm}^B, M \in \text{Term}^A$
(R COAPPLICATION)	$M \succ K \in \text{Term}^{A \otimes B}$	if $K \in \text{CoTerm}^B, M \in \text{Term}^A$
(R SHIFT)	$\mu\alpha.(x * K) \in \text{Term}^B$	if $\alpha \in \text{CoVar}^B, x \in \text{Var}^A, K \in \text{CoTerm}^B$
	$\alpha \in \text{CoTerm}^A$	if $\alpha \in \text{CoVar}^A$
(L APPLICATION)	$M \bowtie K \in \text{CoTerm}^{B \setminus A}$	if $K \in \text{CoTerm}^A, M \in \text{Term}^B$
(R APPLICATION)	$K \bowtie M \in \text{CoTerm}^{A/B}$	if $K \in \text{CoTerm}^A, M \in \text{Term}^B$
(L COABSTR)	$\alpha \odot K \in \text{CoTerm}^{B \otimes A}$	if $\alpha \in \text{CoVar}^B, K \in \text{CoTerm}^A$
(R COABSTR)	$K \odot \alpha \in \text{CoTerm}^{A \otimes B}$	if $\alpha \in \text{CoVar}^B, K \in \text{CoTerm}^A$
(L SHIFT)	$\tilde{\mu}x.(M * \alpha) \in \text{CoTerm}^A$	if $x \in \text{Var}^A, M \in \text{Term}^B, \alpha \in \text{CoTerm}^A$

We formulate the correspondence for Sequent Calculus.

3. Sequents and Co-sequents

Remark By moving from intuitionistic to classical logic, we move from a one-side $(\Gamma \longrightarrow C)$ to a two-side system $(\Gamma \longrightarrow \Delta)$.

Hence, we distinguish sequents (\dagger) and cosequents (\ddagger)

$$(\dagger) \quad \Gamma \longrightarrow \Delta[B] \qquad (\ddagger) \quad \Gamma[A] \longrightarrow \Delta$$

- Γ and Δ are built out \otimes and \oplus , resp. We'll overload \circ .
- A sequent (cosequent) has only one **active** succedent (antecedent) formula.
- Sequents and co-seq. are labeled with proof terms M and coterms K , resp.

$$(\dagger) \quad \Gamma \xrightarrow{M} \Delta[B] \qquad (\ddagger) \quad \Gamma[A] \xrightarrow{K} \Delta$$

- The active formula is unlabeled (since the rule will build its term).
- The passive **antecedent** (**succedent**) formulas are labeled with distinct variables x_i (covariables α_i).

3.1. Axioms

Axioms For the axiomatic case, we distinguish two versions, depending on whether the succedent or the antecedent is the active formula.

$$\frac{}{x : A \xrightarrow{x} A} \text{Ax} \qquad \frac{}{A \xrightarrow{\alpha} \alpha : A} \text{Co-Ax}$$

Intuitively, with the **variables** x we **build** the output term out of the input (we **focus** on the output), whereas with the **co-variable** α we **build** the input term out of the output (we **focus** on the input). I.e.:

left-to-right (\triangleright) vs. right-to-left (\triangleleft).

3.2. Rules of proof ((co)abstraction)

$$\frac{x : B \circ \Gamma \xrightarrow{M} \Delta[A]}{\Gamma \xrightarrow{x \setminus M} \Delta[B \setminus A]} (\setminus R) \qquad \frac{\Gamma[A] \xrightarrow{K} \Delta \circ \alpha : B}{\Gamma[A \otimes B] \xrightarrow{K \otimes \alpha} \Delta} (\otimes L)$$

$$\frac{\Gamma \circ x : B \xrightarrow{M} \Delta[A]}{\Gamma \xrightarrow{M/x} \Delta[A/B]} (/R) \qquad \frac{\Gamma[A] \xrightarrow{K} \alpha : B \circ \Delta}{\Gamma[B \otimes A] \xrightarrow{\alpha \otimes K} \Delta} (\otimes L)$$

Notice that the Grishin interactions are *absorbed* in these rules.

$$\begin{array}{c} \vdots \\ (np \otimes (((np \setminus s)/s) \otimes (np \otimes (np \setminus s)))) \longrightarrow (s \otimes s) \oplus s \\ (s \otimes s) \otimes (np \otimes (((np \setminus s)/s) \otimes (np \otimes (np \setminus s)))) \longrightarrow s \\ \vdots \\ np \otimes (((np \setminus s)/s) \otimes (\underbrace{(s \otimes s) \otimes np}_{QP} \otimes (np \setminus s))) \longrightarrow s \end{array}$$

3.3. Rules of use (function (co)application)

$$\frac{B \xrightarrow{K} \Delta \quad \Delta' \xrightarrow{M} \Gamma[A]}{\Delta' \xrightarrow{M \succ K} \Gamma[(A \otimes B) \circ \Delta]} (\otimes R) \qquad \frac{\Delta \xrightarrow{M} B \quad \Gamma[A] \xrightarrow{K} \Delta'}{\Gamma[\Delta \circ (B \setminus A)] \xrightarrow{M \bowtie K} \Delta'} (\setminus L)$$

$$\frac{B \xrightarrow{K} \Delta \quad \Delta' \xrightarrow{M} \Gamma[A]}{\Delta' \xrightarrow{K \prec M} \Gamma[\Delta \circ (B \otimes A)]} (\otimes R) \qquad \frac{\Delta \xrightarrow{M} B \quad \Gamma[A] \xrightarrow{K} \Delta'}{\Gamma[(A/B) \circ \Delta] \xrightarrow{K \bowtie M} \Delta'} (/L)$$

Note: here we build the term (co-term) of the co-functor (functor), i.e. different from standard labelling of sequents:

$$\frac{\Delta \xrightarrow{M} B \quad \Gamma[y : A] \xrightarrow{N} C}{\Gamma[x : A/B \circ \Delta] \xrightarrow{N[y := x M]} C} (/L)$$

3.4. Commands: Focus

The rules (\Leftarrow) and (\Rightarrow) make it possible to shift the **focus** from antecedent to succedent or vice versa.

$$\frac{\Gamma[A] \xrightarrow{K} \Delta[\beta : B]}{\Gamma[x : A] \xrightarrow{\mu\beta.(x * K)} \Delta[B]} (\Leftarrow) \qquad \frac{\Gamma[x : A] \xrightarrow{M} \Delta[B]}{\Gamma[A] \xrightarrow{\tilde{\mu}x.(M * \beta)} \Delta[\beta : B]} (\Rightarrow)$$

These rules are in fact restricted cuts, where one of the premises is axiomatic (Axiom or Co-Axiom).

$$\frac{x : A \xrightarrow{x} A \quad \Gamma[A] \xrightarrow{K} \Delta[\beta : B]}{\Gamma[x : A] \xrightarrow{x * K} \Delta[\beta : B]} (Cut1) \qquad \frac{\Gamma[x : A] \xrightarrow{M} \Delta[B] \quad B \xrightarrow{\beta} \beta : B}{\Gamma[x : A] \xrightarrow{M * \beta} \Delta[\beta : B]} (Cut2)$$

$$\frac{\Gamma[x : A] \xrightarrow{x * K} \Delta[\beta : B]}{\Gamma[x : A] \xrightarrow{\mu\beta.x * K} \Delta[B]} \qquad \frac{\Gamma[x : A] \xrightarrow{M * \beta} \Delta[\beta : B]}{\Gamma[A] \xrightarrow{\tilde{\mu}x.M * \beta} \Delta[\beta : B]}$$

3.5. Example: Sequents

$$\frac{A \longrightarrow A \quad B \longrightarrow B}{A \circ A \backslash B \longrightarrow B} (\backslash L)$$

$$\frac{A \circ A \backslash B \longrightarrow B}{A \circ A \backslash B \longrightarrow B} (\rightleftharpoons)$$

$$\frac{x : A \xrightarrow{x} A \quad B \xrightarrow{\alpha} \alpha : B}{x : A \circ A \backslash B \xrightarrow{x \ltimes \alpha} \alpha : B} (\backslash L)$$

$$\frac{x : A \circ A \backslash B \xrightarrow{x \ltimes \alpha} \alpha : B}{x : A \circ y : A \backslash B \xrightarrow{\mu \alpha. (y * (x \ltimes \alpha))} B} (\rightleftharpoons)$$

Exercise F1 Try to prove by yourself $A \circ ((A \backslash B) / A \circ A) \vdash B$. **Solution.** Try here, $A \vdash B / (A \backslash B)$ (lifting) and $(B \otimes A) \otimes B \vdash A$ (lowering).

3.6. Labelled Sequents: QP

$$\begin{array}{c}
 \frac{x : np \xrightarrow{x : 1} np \quad s \xrightarrow{\alpha : 2} \alpha : s}{x : np \circ np \backslash s \xrightarrow{1 \times 2 : 3} \alpha : s} (\backslash L) \\
 \frac{x : np \circ y : np \backslash s \xrightarrow{\mu\alpha.(y * 3) : 4} s \quad s \xrightarrow{\beta : 5} \beta : s}{x : np \circ y : np \backslash s \xrightarrow{4 \succ 5 : 6} s \circ \beta : s} (\otimes R) \\
 \frac{x : np \circ y : np \backslash s \xrightarrow{4 \succ 5 : 6} s \circ \beta : s}{np \circ y : np \backslash s \xrightarrow{\tilde{\mu}x.(6 * \gamma) : 7} \gamma : s \circ \beta : s} (\otimes L) \\
 \frac{((s \circ s) \circ np) \circ y : np \backslash s \xrightarrow{\gamma \otimes 7 : 8} \beta : s}{z : ((s \circ s) \circ np) \circ y : np \backslash s \xrightarrow{\mu\beta.(z * 8)} s} (\equiv)
 \end{array}$$

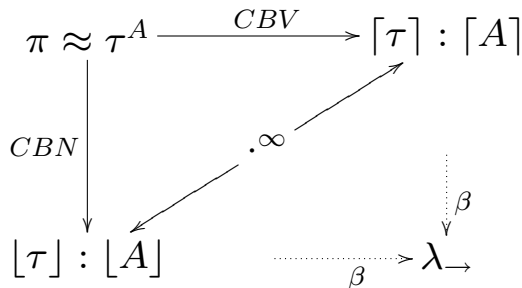
$$\mu\beta. (z * (\gamma \otimes (\tilde{\mu}x. (((\mu\alpha. (y * x \times \alpha)) \succ \beta) * \gamma))))$$

$\tilde{\mu}$ -abs. turns the sentence into a context waiting for the value (np) of its hole.
 Standard practice: to obtain a computational interpretation in λ_{\rightarrow} -terms via CPS.

4. Next Steps: Interpretations

1. We have just seen the Curry-Howard Correspondence ($\pi \approx \tau^A$)
2. We are going to see the Continuation Passing Style interpretation:
 - CBV ($\lceil \tau \rceil : \lceil A \rceil$) or
 - CBN ($\lfloor \tau \rfloor : \lfloor A \rfloor$) regimes

(which are dual \cdot^∞), and transform $\bar{\lambda}\mu\tilde{\mu}$ -terms into λ_{\rightarrow} -terms.



5. Continuation Semantics

Continuation Semantics is used in Programming Language. It has been first applied to Linguistics (independently) by Ph. de Groote and C. Barker. The idea has been further elaborated by C. Barker and C. Shan.

- In a direct semantics, a sentence is seen as denoting a truth value, t .
- In a *continuation semantics*, a sentence is seen as denoting a function of type: $(t \rightarrow R) \rightarrow R$ (Barker '02 and de Groote '01 take $R = t$.)

$$\underbrace{\underbrace{(t \rightarrow R)}_{\text{continuation of } t}}_{\text{computation of } t} \rightarrow R$$

5.1. Continuation Semantics and Proof theory

Direct Semantics: $x_1 : A_1, \dots, x_n : A_n \vdash M : B$

Plotkin '75 (CBV) Hofmann-Streicher '97 (CBN):

- in $\text{CBV}^\triangleright$ the term M is built out of **values**:

$$x_1 : V_{A_1}, \dots, x_n : V_{A_n}, \alpha_1 : K_{A_1}, \dots, \alpha_m : K_{A_m} \vdash M : C_B$$

- in $\text{CBN}^\triangleright$ the term M is built out of **computations**:

$$x_1 : C_{A_1}, \dots, x_n : C_{A_n}, \alpha_1 : K_{A_1}, \dots, \alpha_m : K_{A_m} \vdash M : C_B$$

Instead, $\text{CBV}^\triangleleft : K_B \rightarrow K_A$ vs. $\text{CBN}^\triangleleft : K_B \rightarrow (C_A \rightarrow R)$

Following Herbelin and Curien, for both strategies, we define:

1. Types: the **values**, **continuation**, **computation** of the types we use (constants and (co)functional types)
2. Terms: the CPS transformers from $\bar{\lambda}\mu\tilde{\mu}$ -terms to λ -terms guided by 1.

5.2. Full definitions of Types

For a type A , $\lceil A \rceil$ is a **value** of type A (V_A) (CBV) and $\lfloor A \rfloor$ is a **continuation** of type A (K_A) (CBN). For atoms, $\lfloor p \rfloor = \lceil p^\infty \rceil = p$

$$\begin{aligned}
 \lceil A \backslash B \rceil &= R^{\lceil A \rceil \times R^{\lceil B \rceil}} & R^{\lfloor A \rfloor \times R^{\lfloor B \rfloor}} &= \lfloor B \oslash A \rfloor; \\
 \lceil B / A \rceil &= R^{R^{\lceil B \rceil} \times \lceil A \rceil} & R^{R^{\lfloor B \rfloor} \times \lfloor A \rfloor} &= \lfloor A \oslash B \rfloor; \\
 \lceil B \oslash A \rceil &= \lceil B \rceil \times R^{\lceil A \rceil} & \lfloor B \rfloor \times R^{\lfloor A \rfloor} &= \lfloor A \backslash B \rfloor; \\
 \lceil A \oslash B \rceil &= R^{\lceil A \rceil} \times \lceil B \rceil & R^{\lfloor A \rfloor} \times \lfloor B \rfloor &= \lfloor B / A \rfloor.
 \end{aligned}$$

$$\bowtie \frac{C/D \quad A \otimes B \quad B \oplus A \quad D \oslash C}{D \backslash C \quad B \oslash A \quad A \oplus B \quad C \oslash D} \quad \infty \quad \frac{C/B \quad A \otimes B \quad A \backslash C}{B \oslash C \quad B \oplus A \quad C \oslash A}$$

$$\lceil A \backslash B \rceil = \lfloor (A \backslash B)^\infty \rfloor = \lfloor B^\infty \oslash A^\infty \rfloor$$

$$\lfloor B \oslash A \rfloor = \lceil (B \oslash A)^\infty \rceil = \lceil A^\infty \backslash B^\infty \rceil$$

5.3. Types in CBV vs. CBN: V , K , C

$$\mathbf{CBV}^\triangleright: V_{A_1} \dots V_{A_n} \vdash C_B$$

$$C_A : K_A \rightarrow R$$

$$K_A : V_A \rightarrow R$$

$$V_A : A \text{ — when } A \text{ is atomic}$$

$$V_{A \rightarrow B} : V_A \rightarrow C_B$$

Computation

Continuation

Value

Value

Since, $V_A \rightarrow C_B = V_A \rightarrow (K_B \rightarrow R) \cong (V_A \times K_B) \rightarrow R$, we can take into account directionality of the functional types by using ordered pairs:

$$V_{A \setminus B} = (V_A \times K_B) \rightarrow R \quad \text{vs.} \quad V_{B/A} = (K_B \times V_A) \rightarrow R$$

$$\mathbf{CBN}^\triangleright: C_{A_1} \dots C_{A_n} \vdash C_B$$

$$C_A : K_A \rightarrow R$$

$$K_{(B \otimes A)} : C_A \rightarrow C_B$$

Computation

Continuation

Similarly, we use ordered pairs: $K_{(B \otimes A)} = (C_A \times K_B) \rightarrow R$ vs.

5.4. Full definitions of Types: Examples

$$\begin{aligned}
 [A \setminus B] &= R^{[A] \times R^{[B]}} & R^{[A] \times R^{[B]}} &= [B \oslash A]; \\
 [B / A] &= R^{R^{[B]} \times [A]} & R^{R^{[B]} \times [A]} &= [A \odot B]; \\
 [B \oslash A] &= [B] \times R^{[A]} & [B] \times R^{[A]} &= [A \setminus B]; \\
 [A \odot B] &= R^{[A]} \times [B] & R^{[A]} \times [B] &= [B / A].
 \end{aligned}$$

WORD	TYPE	ALIAS	$[\cdot]$ CBV	$[\cdot]$ CBN
alice, lewis	np		$[np]$	$[np]$
left	$np \setminus s$	iv	$R^{[np] \times R^{[s]}}$	$[s] \times R^{[np]}$
teases	$(np \setminus s) / np$	tv	$R^{R^{[iv]} \times [np]}$	$R^{[np]} \times [iv]$
someone	$s / (np \setminus s)$	sub	$R^{R^{[s]} \times [iv]}$	$R^{[iv]} \times [s]$
someone	$(s \odot s) \odot np$	qp	$R^{[s] \times R^{[s]}} \times [np]$	$R^{R^{[np]} \times R^{[s]} \times R^{[s]}}$

5.5. CPS transformations: Axioms and Shifts

In CBV: $\bar{\lambda}\mu\tilde{\mu}$ **terms** (on the left), and **coterm**s (on the right) are transformed into λ -terms which are **computations** and **continuations**, resp.

$$\begin{array}{c} \frac{}{x : A \xrightarrow{x} \textcolor{violet}{A}} \text{Ax} \qquad \frac{}{\textcolor{violet}{A} \xrightarrow{\alpha} \alpha : A} \text{Co-Ax} \\ C_A \quad [x] = \lambda k.k \ x \qquad K_A \quad [\alpha] = \lambda x.x \ \alpha \end{array}$$

$$\begin{array}{c} \frac{\Gamma[\textcolor{violet}{A}] \xrightarrow{\textcolor{brown}{K}} \Delta[\beta : B]}{\Gamma[x : A] \xrightarrow{\mu\beta.(x * K)} \Delta[\textcolor{violet}{B}]} (\Leftarrow) \qquad \frac{\Gamma[x : A] \xrightarrow{\textcolor{blue}{M}} \Delta[\textcolor{violet}{B}]}{\Gamma[\textcolor{violet}{A}] \xrightarrow{\tilde{\mu}x.(M * \beta)} \Delta[\beta : B]} (\Rightarrow) \\ \\ \frac{x : A \xrightarrow{x} A \quad \Gamma[\textcolor{violet}{A}] \xrightarrow{\textcolor{brown}{K}} \Delta}{\Gamma[A] \xrightarrow{x * K} \Delta} (Cut1) \qquad \frac{\Gamma \xrightarrow{M} \Delta[\textcolor{violet}{B}] \quad B \xrightarrow{\beta} \beta : B}{\Gamma \xrightarrow{M * \beta} \Delta[B]} (Cut2) \\ \\ C_B \quad [\mu\beta.(x * K)] = \lambda\beta.([\textcolor{brown}{K}] \ x) \qquad [\tilde{\mu}x.(M * \beta)] = \lambda x.([\textcolor{red}{M}] \ \beta) \qquad K_A \end{array}$$

5.6. CPS transformations: (Co)Application

$$\frac{\Delta \xrightarrow{M} B \quad \Gamma[A] \xrightarrow{K} \Delta'}{\Gamma[\Delta \circ (B \setminus A)] \xrightarrow{M \bowtie K} \Delta'} (\setminus L)$$

$$K_{B \setminus A} = R^{R^{[B]} \times R^{[A]}} \quad [M \bowtie K] = \lambda k. ([M] \lambda y. (k \langle y, [K] \rangle))$$

$$\frac{B \xrightarrow{K} \Delta \quad \Delta' \xrightarrow{M} \Gamma[A]}{\Delta' \xrightarrow{M \succ K} \Gamma[(A \circ B) \circ \Delta]} (\circ R)$$

$$C_{A \circ B} = R^{R^{[A]} \times R^{[B]}} \quad [M \succ K] = \lambda k. ([M] \lambda x. (k \langle x, [K] \rangle))$$

Note, CPS interpretation shows that application and co-application boil down to the same operation semantically.

5.7. CPS transformations: (co)Abstraction

$$\frac{y : B \circ \Gamma \xrightarrow{M} \Delta[A]}{\Gamma \xrightarrow{y \setminus M} \Delta[B \setminus A]} (\setminus R)$$

$$C_{B \setminus A} = R^{R^{R[B] \times R[A]}} \quad [y \setminus M] = \lambda k. (\textcolor{brown}{k} \lambda \langle \textcolor{blue}{y}, \alpha \rangle. ([M] \alpha))$$

$$\frac{\Gamma[A] \xrightarrow{K} \Delta \circ \alpha : B}{\Gamma[A \oslash B] \xrightarrow{K \oslash \alpha} \Delta} (\oslash L)$$

$$K_{A \oslash B} = R^{[A] \times R^{[B]}} \quad [K \oslash \alpha] = \lambda \langle \textcolor{blue}{x}, \alpha \rangle. ([K] \textcolor{blue}{x})$$

The CBN regime is the composition of CBV and arrow reversal: $[\cdot] \triangleq [\cdot]^\infty$.

5.8. Summary of all the CPS transformations

$[\cdot] : \text{Terms} \rightarrow \text{Computations}$ and $\text{Co-terms} \rightarrow \text{Continuations}$

A	$[x] = \lambda k.k \ x$	$x : A$
$B \setminus A$	$[x \setminus M] = \lambda k.(k \ \lambda \langle x, \beta \rangle. [M] \ \beta)$	$x : B, M : A$
A/B	$[M / x] = \lambda k.(k \ \lambda \langle \beta, x \rangle. [M] \ \beta)$	$x : B, M : A$
$A \odot B$	$[M \succ K] = \lambda k.([M] \ \lambda y.(k \ \langle y, [K] \rangle))$	$M : A, K : B$
$B \otimes A$	$[K \prec M] = \lambda k.([M] \ \lambda y.(k \ \langle [K], y \rangle))$	$M : A, K : B$
B	$[\mu \alpha.(x * K)] = \lambda \alpha.([K] \ x)$	$\alpha : B, x, K : A$
A	$[\alpha] = \lambda x.\alpha \ x$	$\alpha : A$
$B \otimes A$	$[\alpha \otimes K] = \lambda \langle \alpha, x \rangle.([K] \ x)$	$\alpha : B, K : A$
$A \odot B$	$[K \odot \alpha] = \lambda \langle x, \alpha \rangle.([K] \ x)$	$\alpha : B, K : A$
$B \setminus A$	$[M \bowtie K] = \lambda k.([M] \ \lambda x.(k \ \langle x, [K] \rangle))$	$M : B, K : A$
A/B	$[K \bowtie M] = \lambda k.([M] \ \lambda x.(k \ \langle [K], x \rangle))$	$M : B, K : A$
A	$[\tilde{\mu} x.(M * \alpha)] = \lambda x.([M] \ \alpha)$	$x : A, \alpha, M : B$

6. Proofs and Refutations

Proof: A implies B		
$A \vdash_{\text{CBV}^\triangleright} B$	\bowtie	$A \vdash_{\text{CBV}^\triangleleft} B$
$V_A \rightarrow C_B$		$K_B \rightarrow K_A$
$(V_A \times K_B) \rightarrow R = [A \setminus B]$	\bowtie	$[B/A] = (K_B \times V_A) \rightarrow R$
Refutation: A and $\neg B$		
$A \vdash_{\text{CBN}^\triangleleft} B$	\bowtie	$A \vdash_{\text{CBN}^\triangleright} B$
∞		∞
$B^\infty \vdash_{\text{CBV}^\triangleright} A^\infty$		$B^\infty \vdash_{\text{CBV}^\triangleleft} A^\infty$
$K_B \rightarrow (C_A \rightarrow R)$		$C_A \rightarrow C_B$
$(K_B \times C_A) \rightarrow R = [A \oslash B]$	\bowtie	$[B \oslash A] = (C_A \times K_B) \rightarrow R$

6.1. CBN is dual to CBV

The CBN interpretation of a given theorem, e.g. of $A \circ A \setminus B \vdash B$ is obtained into possible ways:

(a) by dualizing the proof: $\Gamma \vdash \Delta$ becomes $\Delta^\infty \vdash \Gamma^\infty$

$$(A \circ A \setminus B \vdash B)^\infty \text{ which is } B^\infty \vdash B^\infty \odot A^\infty \circ A^\infty.$$

(b) by dualizing the CBV proof term (recall, $\lfloor \cdot \rfloor$ (CBN) vs. $\lceil \cdot \rceil$ (CBV).)

$$\lfloor M \rfloor = \lceil M^\infty \rceil.$$

The dualities we discussed for the type system extend to the term language:

$$\begin{array}{llll}
 \textcolor{blue}{x}^\infty & = & \textcolor{orange}{\alpha} & \textcolor{orange}{\alpha}^\infty = \textcolor{blue}{x}; \\
 (x \setminus M)^\infty & = & M^\infty \odot \alpha & (K \odot \alpha)^\infty = x \setminus K^\infty; \\
 (M / x)^\infty & = & \alpha \odot M^\infty & (\alpha \odot K)^\infty = K^\infty / x; \\
 (M \succ K)^\infty & = & K^\infty \ltimes M^\infty & (M \ltimes K)^\infty = K^\infty \succ M^\infty; \\
 (K \prec M)^\infty & = & M^\infty \rtimes K^\infty & (K \rtimes M)^\infty = M^\infty \prec K^\infty; \\
 (\mu\beta.(x * K))^\infty & = & \tilde{\mu}y.(K^\infty * \alpha) & (\tilde{\mu}y.(M * \alpha))^\infty = \mu\beta.(x * M^\infty).
 \end{array}$$

6.2. Example: CBV vs. CBN proofs

$$\begin{array}{c}
 \text{CBV} \\
 \frac{\frac{x : A \xrightarrow{x} A \quad B \xrightarrow{\alpha} \alpha : B}{x : A \circ A \setminus B \xrightarrow{x \ltimes \alpha} \alpha : B} (\setminus L)}{x : A \circ y : A \setminus B \xrightarrow{\mu\alpha.(y * (x \ltimes \alpha))} B} (\Leftarrow)
 \end{array}$$

$$\begin{array}{c}
 \text{CBN} \\
 \frac{\frac{A \xrightarrow{\alpha} \alpha : A \quad y : B \xrightarrow{y} B}{y : B \xrightarrow{y \prec \alpha} B \oslash A \circ \alpha : A}}{B \xrightarrow{\tilde{\mu}y.((y \prec \alpha) * \beta)} \beta : B \oslash A \circ \alpha : A}
 \end{array}$$

Recall that in the CBN derivation the atoms A and B are actually standing for their duals, hence α and y are of type A^∞ and B^∞ , resp.

6.3. Example: CBV vs. CBN terms

(b) Recall: $\lfloor M \rfloor = \lceil M^\infty \rceil$.

$$\begin{array}{l}
 \text{CBV} \quad \frac{\frac{x : A \xrightarrow{x} A \quad B \xrightarrow{\alpha} \alpha : B}{x : A \circ A \setminus B \xrightarrow{x \ltimes \alpha} \alpha : s} (\setminus L)}{x : A \circ y : A \setminus B \xrightarrow{\mu\alpha.(y * (x \ltimes \alpha))} B} (\Leftarrow) \\
 \\
 \text{CBN} \quad \begin{aligned}
 \lfloor \mu\alpha.(y * (x \ltimes \alpha)) \rfloor &= \lceil (\mu\alpha.(y * (x \ltimes \alpha)))^\infty \rceil \\
 &= \lceil \tilde{\mu}y.((x \ltimes \alpha)^\infty * y^\infty) \rceil \\
 &= \lceil \tilde{\mu}y.((\alpha^\infty \prec x^\infty) * \beta) \rceil \\
 &= \lceil \tilde{\mu}y.((y \prec \alpha) * \beta) \rceil
 \end{aligned}
 \end{array}$$

Recall, in CBN induction starts from K : y , α , and β .

7. Meaning in "Parsing as deduction"

$$\begin{array}{ccc} w_1 & \cdots & w_n \\ \vdots & & \vdots \\ x_1 : A_1 & \cdots & x_n : A_n \end{array} \vdash M : B$$

$\underbrace{\hspace{10em}}_{\Gamma}$

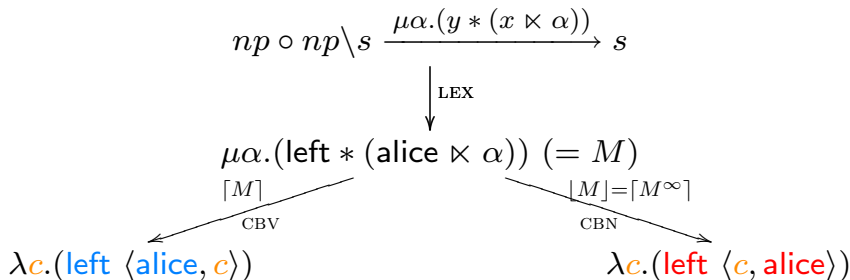
The grammatical logic as a PROGRAMMING LANGUAGE: read the proof of the derivation $A_1, \dots, A_n \vdash B$ as an instruction for the assembly of a meaning program M with input parameters x_1, \dots, x_n .

Direct S. $x_1 : A_1, \dots, x_n : A_n \vdash M : B$

CBV[▷] $\textcolor{blue}{x}_1 : \textcolor{blue}{V}_{A_1}, \dots, \textcolor{blue}{x}_n : \textcolor{blue}{V}_{A_n}, \alpha_1 : \textcolor{brown}{K}_{A_1}, \dots, \alpha_m : \textcolor{brown}{K}_{A_m} \vdash \textcolor{red}{M} : \textcolor{red}{C}_B$

CBN[▷] $\textcolor{red}{x}_1 : \textcolor{red}{C}_{A_1}, \dots, \textcolor{red}{x}_n : \textcolor{red}{C}_{A_n}, \alpha_1 : \textcolor{brown}{K}_{A_1}, \dots, \alpha_m : \textcolor{brown}{K}_{A_m} \vdash \textcolor{red}{M} : \textcolor{red}{C}_B$

7.1. Two interpretations in Continuation Semantics



- CBV:
 - "left" denotes in the domain of functions: $(V_{np} \times K_s) \rightarrow R$;
 - "alice" denotes in the domain of V_{np} .
- CBN:
 - "left" denotes in the domain of functions: $(K_s \times C_{np}) \rightarrow R$;
 - "alice" denotes in the domain of C_{np} .

7.2. CPS transformation: Example (CBV)

$$\begin{aligned} \lceil \mu\alpha.(\text{left} * (\text{alice} \ltimes \alpha)) \rceil &= \lambda\alpha. \lceil \text{alice} \ltimes \alpha \rceil \text{ left} \\ &= \lambda\alpha. (\lambda k. \lceil \text{alice} \rceil \lambda x. (k \langle x, \lceil \alpha \rceil \rangle)) \text{ left} \\ &= \lambda\alpha. (\lceil \text{alice} \rceil \lambda x. (\text{left} \langle x, \lceil \alpha \rceil \rangle)) \\ &= \lambda\alpha. ((\lambda k. k \text{ alice}) \lambda x. (\text{left} \langle x, \alpha \rangle)) \\ &= \lambda\alpha. ((\lambda x. (\text{left} \langle x, \alpha \rangle)) \text{ alice}) \\ &= \lambda\alpha. ((\text{left} \langle \text{alice}, \alpha \rangle)) \end{aligned}$$

7.3. CPS transformation: Example (CBN)

$$\begin{aligned} \llbracket \mu\alpha.(\text{left} * (\text{alice} \times \alpha)) \rrbracket &= \lceil (\mu\alpha.(\text{left} * (\text{alice} \times \alpha)))^\infty \rceil \\ &= \lceil \tilde{\mu}y.((y \succ \text{alice}) * \text{left}) \rceil \\ &= \lambda y.(\lceil y \succ \text{alice} \rceil) \text{ left}) \\ &= \lambda y.(\lambda k.(\lceil y \rceil \lambda z.(k \langle z, \lceil \text{alice} \rceil \rangle))) \text{ left}) \\ &= \lambda y.(\lceil y \rceil \lambda z.(\text{left} \langle z, \lceil \text{alice} \rceil \rangle)) \\ &= \lambda y.((\lambda k.k \ y) \lambda z.(\text{left} \langle z, \lceil \text{alice} \rceil \rangle)) \\ &= \lambda y.((\lambda z.(\text{left} \langle z, \lceil \text{alice} \rceil \rangle)) \ y) \\ &= \lambda y.((\text{left} \langle y, \text{alice} \rangle)) \end{aligned}$$

Recall: in CBN the induction starts from continuations.

Exercise F2 Try by your self “Alice teases Lewis”. Steps: (a) sequent, (b) labelled sequent, (c) CBV, (d) CBN.

8. Scope and Binding!

Problem Recall, a QP is an in situ binder:

1. syntactically occupies the position of a phrase of type np .
2. semantically: it **binds** an np -type variable in that position and can take **scope** over clausal level (scope domain) higher than where it occurs.

Solution LG gives us a way to express type assignment, $(s \otimes s) \otimes np$ that overcomes the problems of a Lambek assignment $s/(np \backslash s)$.

- Simple QP context: $(s \otimes s) \otimes np$ versus $s/(np \backslash s)$.
- Multiple QP examples:
 - scope ambiguity: local
 - local versus non-local construal

Exercises G1 Try the simple QP context: “Someone left”: (a) labelled sequent, (b) CBV and CBN.

8.1. Scope ambiguity, local

Type uniformity: $(s \otimes s) \otimes np$ fits both the subject and the object role. The scope ambiguity arises from the nondeterministic choice between the **subject** or **object** ($\otimes L$) rule as the last step of the derivation.

E.g., "Everyone teases someone"

$$\frac{\vdots}{((s \otimes s) \otimes np) \circ (tv \circ ((s \otimes s) \otimes np)) \vdash s} (\otimes L)$$

$$\begin{aligned} & \lambda c.(\text{evr } \lambda \langle q, y \rangle.(\text{sm } \lambda \langle p, z \rangle.(y \langle c, \lambda c'.(z \langle c', \lambda c''.(\text{teases } \langle p, \langle c'', q \rangle \rangle)))))) \\ & \lambda c.(\text{sm } \lambda \langle p, z \rangle.(\text{evr } \lambda \langle q, y \rangle.(z \langle c, \lambda c'.(y \langle c', \lambda c''.(\text{teases } \langle p, \langle c'', q \rangle \rangle)))))) \end{aligned}$$

Exercise G2 Try the derivation in CBV and CBN your self. Tomorrow we will look into the solutions. (a) labelled sequents, (b) CBN.

8.2. Local vs. non-local construal

The ambiguity arises here from the fact that the QP can non-deterministically select the **embedded** or the **main** clause as its scope domain: **local** versus **non-local** scope readings. Eg., “Alice thinks someone left”

$$\frac{\vdots}{(np \circ (iv / s \circ ((s \otimes s) \otimes np) \circ iv)) \vdash s} (/L)$$

$$\frac{\vdots}{(np \circ (iv / s \circ ((s \otimes s) \otimes np) \circ iv)) \vdash s} (\otimes L)$$

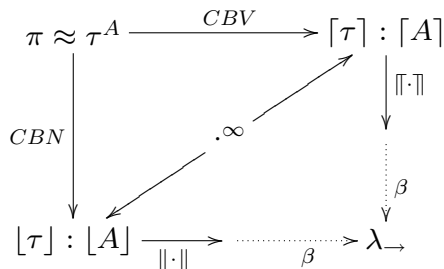
$\lambda c. (\text{thinks } \langle \lambda c'. (\text{sm } \lambda \langle q, y \rangle. (y \langle c', \lambda c''. (\text{left } \langle c'', q \rangle))) \rangle, \langle c, a \rangle))$

$\lambda c. (\text{sm } \lambda \langle q, y \rangle. (y \langle c, \lambda c'. (\text{thinks } \langle \lambda c''. (\text{left } \langle c'', q \rangle), \langle c', a \rangle \rangle)))$

Exercise G3 Try the derivation in CBV and CBN your self. Tomorrow we will look into the solutions. (a) labelled sequents, (b) CBN.

9. What have you learned today?

- We have seen:
 1. Curry-Howard Interpretation ($\pi \approx \tau^A$)
 2. Continuation Passing Style interpretations: CBV ($[\tau] : [A]$) or CBN ($[\tau] : [A]$) regimes (which are dual)
- Tomorrow we will see:
 3. Connection with Montague style interpretation via lifting of the lexicon constants to the CBV ($[\tau] : [A]$) or the CBN ($[\tau] : [A]$) level.



10. More to Explore

In the course materials you find

- Ch. 3 covers the material of today session.
- Ch. 2 gives direct CPS interpretations of the combinator derivations.

The References of Ch.3 contain useful hints for further readings, especially:

- Curien and Herbelin 2000
- Selinger 2001
- Wadler 2003