Symmetric categorial grammar

Monday

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Abstract

Lambek-style typelogical grammar is more attractive than rule-based alternatives in that the Lambek calculi have full support for hypothetical reasoning. Ironically, the hypothetical reasoning component of the original Lambek grammars is deficient: they cannot adequately characterize 'visibility' of hypotheses in their contextual environment. We discuss the extensions of Lambek calculus proposed by Grishin (1983) and show how they provide an elegant solution to Lambek's problems, based on symmetry between implications and coimplications, and on structure-preserving interactions between them.

Contents

1. The Lambek program

THE MATHEMATICS OF SENTENCE STRUCTURE*

JOACHIM LAMBEK, McGill University

The definitions [of the parts of speech] *are very far from having attained the degree of exactitude found in Euclidean geometry.*

—Otto Jespersen, 1924.

1. Introduction. The aim of this paper is to obtain an effective rule (or algorithm) for distinguishing sentences from nonsentences, which works not only for the formal languages of interest to the mathematical logician, but also for natural languages such as English, or at least for fragments of such languages. An attempt to formulate such an algorithm is implicit in the work of Ajdukiewicz.† His method, later elaborated by Bar-Hillel [2], depends on a kind of arithmetization of the so-called *parts of speech*, here called *syntactic types*.‡

from: Lambek 1958

2. Course outline

Slogans Grammar = logic; parsing = deduction

$$\begin{array}{cccc} A_1 & \cdots & A_n & \Rightarrow B \\ w_1 & \cdots & w_n & \end{array}$$

Requirements for a grammar logic

- It should be able to discriminate linguistically relevant aspects of grammatical organization: multiplicity, word order, phrase structure
- Syntax-semantics interface: derivations as instructions for meaning assembly

Starting point Limitations of the original 1958 calculus (known as **L**).

- Too weak: discontinuous dependencies (extraction, infixation)
- Too strong: insensitive to constituent structure (compare **NL** of 1961)

Related work addressing these problems

- Discontinuous Lambek calculus (Morrill, . . .).
  - Enriched ontology: strings, split strings
- Abstract Categorial Grammars (De Groote, Muskens, . . .).
  - Higher-order linear signatures; abstract vs concrete syntax.
- Continuation semantics (Barker, Shan, . . .).
  - Direct compositionality; single derivation, multiple evaluation strategies
- Modalities for structural control (Utrecht school)
  - Minimal logic for syntax-semantics interface;
  - meaning-preserving structural deformations.

Common feature of these approaches: derivability is an asymmetric relation $A_1, \ldots, A_n \Rightarrow B$ (the "intuitionistic" restriction)

Our approach We restore the symmetry.

- **LG** = symmetric **NL** + structure preserving interaction principles
- Symmetry: $A_1 \otimes \cdots \otimes A_n \Rightarrow B_1 \oplus \cdots \oplus B_m$
- Interaction: respecting word order and phrase structure

LG stands for Lambek-Grishin calculus, cf Grishin 1983

Об одном обобщении системы Айдукевича–Ламбека

Methodological remark Throughout the course, we put equal emphasis on model-theoretic and proof-theoretic aspects.

- Model-theoretic side: completeness
- Proof-theoretic side: decidability

3. Sessions

Monday Introducing Lambek-Grishin calculus **LG**. Syntax; frame semantics; completeness; decidability.

Tuesday Curry-Howard interpretation. Terms for **LG** derivations. Continuation-passing-style interpretation.

Wednesday Lexical versus derivational semantics. Case study: scope construal.

Thursday Relations between types: derivability versus similarity. Case studies: extraction, crossed dependencies.

Friday Residuated and Galois connected unary modalities: syntax and semantics. Discussion.

Sign up at symcg.pbwiki.com for slides, extra material, discussion.

Part I
Residuation and structural control

4. Vocabulary

The parts of speech are turned into

- formulas: logical perspective
- types: computational perspective

A, B	$::=$	$p \mid$	atoms: s sentence, np noun phrase, ...
		$\Diamond A \mid \Box A \mid$	features: key, lock
		$A \otimes B \mid A/B \mid B\backslash A$	product, right vs left division

5. Semantics

Lambek calculus has two kinds of semantics:

- Structural semantics:
  - interpretation on relational frames (Kripke)
  - categorial grammar as a modal logic of "grammatical structures"
- Computational semantics: Curry-Howard interpretation of derivations
  - Division types as functions: $D_{A\backslash B} = D_{B/A} = D_B^{D_A}$
  - Rules of use for $/, \backslash$: functional application; rules of proof: λ abstraction
  - Similar pattern for other connectives

Remark We discuss the Curry-Howard interpretation in Tuesday's lecture.

6. Structural semantics

Modal logic: 'logic of structures'.

Logic of language: grammatical structures.

- Frames $F = \langle W, R^2_\Diamond, R^3_\otimes \rangle$
  - W: 'signs', linguistic resources, expressions
  - $R^3_\otimes$: 'Merge', grammatical composition
  - $R^2_\Diamond$: 'feature checking', structural control
- Models $\mathcal{M} = \langle F, V \rangle$
- Valuation $V : \mathrm{PROP} \mapsto \mathcal{P}(W)$: types as sets of expressions

Remark Our language is purely modal: in this course we focus on composition. The lattice operations can be added, if desired, as originally in Lambek 61.

7. Interpretation of the constants

Inverse duality $\otimes$ and $\Diamond$ as existential multiplicative modalities; slashes and $\Box$ as duals with respect to the rotations of $R_\otimes$ and $R_\Diamond$

$$\begin{array}{lll}
x \Vdash \Diamond A & \text{iff} & \exists y.R_\Diamond xy \text{ and } y \Vdash A \\
y \Vdash \Box B & \text{iff} & \forall x.R_\Diamond xy \text{ implies } x \Vdash B \\
& & \\
x \Vdash A \otimes B & \text{iff} & \exists yz.R_\otimes xyz \text{ and } y \Vdash A \text{ and } z \Vdash B \\
y \Vdash C/B & \text{iff} & \forall xz.(R_\otimes xyz \text{ and } z \Vdash B) \text{ implies } x \Vdash C \\
z \Vdash A\backslash C & \text{iff} & \forall xy.(R_\otimes xyz \text{ and } y \Vdash A) \text{ implies } x \Vdash C
\end{array}$$

Compare For $\Diamond, \Box$: $\langle F\rangle, [P]$ in minimal temporal logic; for $\otimes$ and its residuals: fusion in relevant logics.

$$p \not\vdash \langle F\rangle p \qquad [P]p \not\vdash p$$

$$\langle F\rangle[P]p \vdash p \vdash [P]\langle F\rangle p$$

8. The minimal grammar logic

The minimal grammar logic is given by the preorder laws for derivability (reflexivity $A \longrightarrow A$) and transitivity: from $A \longrightarrow B$ and $B \longrightarrow C$, deduce $A \longrightarrow C$), together with the residuation laws below.

Residuation laws relating pairs of opposites (inverse duals):

$$
\begin{array}{lrcl}
(\textsc{res-1}) & \Diamond A \longrightarrow B & \text{iff} & A \longrightarrow \Box B \\[6pt]
(\textsc{res-l}) & A \otimes B \longrightarrow C & \text{iff} & A \longrightarrow C/B \\
(\textsc{res-r}) & A \otimes B \longrightarrow C & \text{iff} & B \longrightarrow A\backslash C
\end{array}
$$

Some theorems $\Diamond\Box s \longrightarrow s \longrightarrow \Box\Diamond s$ (subtyping)

$$np \longrightarrow s/(np\backslash s) \text{ (lifting)} \qquad (s/(np\backslash s))\backslash s \longrightarrow np\backslash s \text{ (lowering)}$$

And some non-theorems

$$np \not\longrightarrow s/(s/np) \qquad s/(np\backslash s) \not\longrightarrow (s/np)/((np\backslash s)/np) \text{ (Geach)}$$

9. Completeness, invariants

Soundness, completeness (Došen 92) In the minimal grammar logic

$$\vdash A \longrightarrow B \qquad \text{iff} \qquad \forall F, V,\ V(A) \subseteq V(B)$$

Invariants The minimal grammar logic puts no constraints on the interpretation of Merge/Check:

- the laws of the base logic hold no matter what the structural particularities of individual languages are
- these laws then capture grammatical invariants

String language generated A grammar G is an assignment of a finite number of types to the vocabulary items. $L(G)$ is $L(G, s)$ for start symbol s. We say $w_1 \cdots w_n \in L(G, B)$ iff $\exists A_i, X$ such that $(w_i, A_i) \in G$, X is a $\Diamond, \otimes$ tree with yield $A_1, \ldots, A_n$ and $\vdash X \rightarrow B$.

10. Emergence of grammatical notions

Grammatical notions and their properties, rather than being postulated, emerge from the type structure. Some examples:

- Subcategorization, valency. Intransitive $np\backslash s$, transitive $(np\backslash s)/np$, ditransitive $((np\backslash s)/np)/np$, etc
- Case. Subject $s/(np\backslash s)$, direct object $((np\backslash s)/np)\backslash(np\backslash s)$, prepositional object $(pp/np)\backslash pp$, etc
- Complements versus modifiers. Compare exocentric A/B with $A \neq B$ versus endocentric A/A categories. Optionality of the latter follows.
- Embryonic form of binding and filler-gap dependencies. Nested implications $C/(A\backslash B)$ signal withdrawal (binding) of a gap hypothesis A in a domain B.

11. Some examples

	SU	$\otimes$	(*TV*	$\otimes$	*OBJ*)
Leopold likes Molly	np		$(np\backslash s)/np$		np
He likes Molly	$s/(np\backslash s)$		$(np\backslash s)/np$		np
Leopold likes her	np		$(np\backslash s)/np$		$((np\backslash s)/np)\backslash(np\backslash s)$
Stephen hates himself	np		$(np\backslash s)/np$		$((np\backslash s)/np)\backslash(np\backslash s)$
Noone hates himself	$s/(np\backslash s)$		$(np\backslash s)/np$		$((np\backslash s)/np)\backslash(np\backslash s)$
Who likes Molly?	$wh/(np\backslash s)$		$(np\backslash s)/np$		np
A solution is badly needed	$s/(np\backslash s)$			$(s/(np\backslash s))\backslash s$	

The latter example has two derivations — hence two readings (de dicto, de re). But . . .

12. Problems with the minimal system

- The minimal grammar logic (unlike cfg) gives equal prominence to
  - putting phrases together (modus ponens, application)
  - taking phrases apart (hypothetical reasoning, abstraction)
- But: the characterization of 'visible' hypotheses is deficient

type	meaning (Curry-Howard)
$C/(A\backslash B)$	$(M\ \lambda x^A.N^B)^C$

Which positions can the A hypothesis (gap) occupy? Two kinds of problems:

- Syntactic displacement. Example: wh "movement"
- Non-local semantic construal. Example: wh "in situ"

13. Illustration

Syntactic displacement The A-type hypothesis occurs internally within the domain of type B. Compare:

Leopold knows what$_{wh/(np\backslash s)}$ __$_{np}$ irritates Molly
Leopold knows what$_{wh/(np???s)}$ Molly suggested __$_{np}$ to Mulligan

Non-local semantic construal Converse of the above: the functor $C/(A\backslash B)$ occupies the structural position where in fact an A-type expression is needed, and realizes its semantic effect at a higher structural level.

Compare E "What did Bloom buy __$_{np}$?" with Japanese

$$\overset{\text{Bloom} - \textsc{nom}}{((\underbrace{nom}_{\text{Bloom-ga}}} \otimes \overset{\text{what} - \textsc{acc}}{\underbrace{((wh???(np???q))}_{\text{nani-o}}} \otimes \overset{\text{bought}}{\underbrace{(acc\backslash(nom\backslash s))}_{\text{katta}}})) \otimes \overset{\text{Q}}{\underbrace{(s\backslash q)}_{\text{ka}}}) \rightarrow wh$$

14. Global structural rules

Initial attempts to address these problems relied on global structural rules for $\otimes$

- associativity: forget constituent structure
- commutativity: forget word order

$\rightsquigarrow$ too crude for linguistic purposes.

$$\begin{tikzcd}
 & \textbf{LP} & \\
\textbf{NLP} \arrow[ur] & & \textbf{L} \arrow[ul] \\
 & \textbf{NL} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

15. Constants for structural control

Modalities $\Diamond$, $\Box$ provide new forms of expressivity:

- Logical: subtyping via $\Diamond\Box A \rightarrow A \rightarrow \Box\Diamond A$
- Structural: controlled (instead of global) structural reasoning, anchored in lexical type assignment.

Example left/right extraction; converses: infixation (Vermaat 2006)

$$
\begin{array}{ll}
(P1)\ \Diamond A \otimes (B \otimes C) \rightarrow (\Diamond A \otimes B) \otimes C & (C \otimes B) \otimes \Diamond A \rightarrow C \otimes (B \otimes \Diamond A)\ (P3) \\
(P2)\ \Diamond A \otimes (B \otimes C) \rightarrow B \otimes (\Diamond A \otimes C) & (C \otimes B) \otimes \Diamond A \rightarrow (C \otimes \Diamond A) \otimes B\ (P4)
\end{array}
$$

Theorem (Kurtonina/Moortgat 97) The systems **(N)L(P)** can mutually simulate each other in terms of $\Diamond, \Box$ embedding translations.

16. Survey of results

- Some PhD theses @ OTS
  - Kurtonina 1995, Frames & Labels.
  - Moot 2002, Proof Nets for Linguistic Analysis.
  - Bernardi 2002, Reasoning with Polarity in Categorial Type Logic.
  - Vermaat 2006, The logic of variation. A cross-linguistic account of wh-question formation in type logical grammar.
  - Capelletti 2007. Parsing with structure preserving categorial grammars.
- Moortgat 1997, Categorial type logics. Handbook of Logic and Language, Chapter 2. Elsevier/MIT Press.

Part II. Symmetry

17. A diagnosis and a cure

We summarize the problem with phenomena of extraction and infixation (in situ construal) in the Lambek framework.

Diagnosis Lambek calculi obey an “intuitionistic” restriction: sequents have a single formula on the right, multiple formulas on the left. Extraction ($\uparrow$) and infixation ($\downarrow$) have a natural ($\uparrow R$) and ($\downarrow L$) rule; matching ($\uparrow L$) and ($\downarrow R$) cannot be formulated: there is no context right of $\Rightarrow$.

$$\frac{\Gamma, B, \Gamma' \Rightarrow A}{\Gamma, \Gamma' \Rightarrow A \uparrow B} \uparrow R \qquad \frac{\Delta, \Delta' \Rightarrow B \quad \Gamma, A, \Gamma' \Rightarrow C}{\Gamma, \Delta, B \downarrow A, \Delta', \Gamma' \Rightarrow C} \downarrow L$$

Cure We remove the asymmetry, following Grishin 1983

- a symmetric base logic: $\otimes$ (fusion) versus $\oplus$ (fission)
- extended with structure-preserving interaction principles

18. Vocabulary

$A, B \;::=\; p \mid$	atoms: s sentence, np noun phrase, ...
$A \otimes B \mid B \backslash A \mid A / B \mid$	product, left vs right division
$A \oplus B \mid A \oslash B \mid B \obslash A$	coproduct, right vs left difference

How To Say It in LG

A/B	"A over B"
$B \backslash A$	"B under A"
$A \oslash B$	"A minus B"
$B \obslash A$	"B from A"

Remark Today we restrict attention to the binary vocabulary. The symmetric extension with unary connectives is straightforward; see Friday's lecture.

19. Lambek-Grishin calculus

We arrive at **LG** in two steps:

- the minimal symmetric system $\mathbf{LG}_\emptyset$
- extension with interaction principles

$\mathbf{LG}_\emptyset$ consists of

- preoder axioms: $A \longrightarrow A$; from $A \longrightarrow B$ and $B \longrightarrow C$ conclude $A \longrightarrow C$
- (dual) residuation principles

$$A \longrightarrow C/B \quad \text{iff} \quad A \otimes B \longrightarrow C \quad \text{iff} \quad B \longrightarrow A \backslash C$$

$$B \varobslash C \longrightarrow A \quad \text{iff} \quad C \longrightarrow B \oplus A \quad \text{iff} \quad C \oslash A \longrightarrow B$$

20. Through the Looking Glass

$\mathbf{LG}_\emptyset$ exhibits two types of symmetry: $\bowtie$ is order-preserving, ∞ is order-reversing:

$$A^{\bowtie} \longrightarrow B^{\bowtie} \quad \text{iff} \quad A \longrightarrow B \quad \text{iff} \quad B^{\infty} \longrightarrow A^{\infty}$$

$$\bowtie \quad \begin{array}{cccc} C/D & A \otimes B & B \oplus A & D \obslash C \\ \hline\hline D\backslash C & B \otimes A & A \oplus B & C \oslash D \end{array} \qquad\qquad \infty \quad \begin{array}{ccc} C/B & A \otimes B & A\backslash C \\ \hline\hline B \obslash C & B \oplus A & C \oslash A \end{array}$$

$\rightsquigarrow$ theorems form quartets:

$$\begin{tikzcd} (B \oslash A) \obslash B \longrightarrow A \arrow[r, leftrightarrow, dotted] \arrow[d, leftrightarrow, dotted] & B \oslash (A \obslash B) \longrightarrow A \arrow[d, leftrightarrow, dotted, "\infty"] \\ A \longrightarrow B/(A\backslash B) \arrow[r, leftrightarrow, dotted, "\bowtie"'] & A \longrightarrow (B/A)\backslash B \end{tikzcd}$$

21. The need for interaction

Motivation Moving to symmetric $\mathbf{LG}_{\emptyset}$ by itself doesn't help much to address the inadequacies of Lambek calculus. Suppose we want to assign a type $B \oslash C$ to a word. When we use this word in building a phrase, $\oslash$ is trapped in its $\otimes$ context:

$$A_1 \otimes \cdots A_i \otimes (B \oslash C) \otimes A_{i+2} \cdots A_n \vdash D$$

Structure preservation Which properties of grammatical organization do we want our interaction principles to preserve?

- word order: interaction should respect the non-commutativity of $\otimes/\oplus$
- phrase structure: interaction should respect their non-associativity

Grishin 1983 provides the recipe to compute all combinatorial possibilities that satisfy the structure preservation requirement.

22. Interaction principles

Notation For $* \in \{/, \otimes, \backslash, \oslash, \oplus, \oslash\}$, we write $a^{?*}b \stackrel{df}{=} b * a$ and $a^{*?}b \stackrel{df}{=} a * b$.

The matrix Let

$$\begin{array}{rcl} M & = & \{ \quad ?\otimes, \quad \otimes?, \quad \oslash?, \quad ?\oslash \quad \} \\ \Lambda & = & \{ \quad \oplus?, \quad ?\oplus, \quad ?/, \quad \backslash? \quad \} \quad = \quad M^{\infty} \end{array}$$

$M \times \Lambda$ defines 16 extensions of $\mathbf{LG}_{\emptyset}$ in terms of postulates of the form

$$a^{\mu}b^{\lambda}c \longrightarrow b^{\lambda}a^{\mu}c$$

23. Interaction principles (cont'd)

	$\oplus ?$	$? \oplus$	$?/$	$\backslash ?$
$? \otimes$	I_1	I_2		
$\otimes ?$	I_4	I_3		
$\obslash ?$			J_1	J_2
$? \oslash$			J_4	J_3

- Eight of these are same-sort associativities and commutativities: they violate structure preservation.
- The remaining eight relate the $\otimes$ and the $\oplus$ families; they are structure preserving.

Example

$$(I_3) \quad a \otimes (c \oplus b) \longrightarrow (a \otimes c) \oplus b \qquad\qquad (b\backslash c) \oslash a \longrightarrow b\backslash(c \oslash a) \quad (J_3)$$

24. Class I versus Class IV

The following equivalences are easily established (see Exercises).

$$
\begin{array}{llcll}
(I_3) & b \otimes (c \oplus a) \rightarrow (b \otimes c) \oplus a & \text{iff} & (a \otimes c) \oslash b \rightarrow a \otimes (c \oslash b) & (I_3)' \\
(J_3) & (b \backslash c) \oslash a \rightarrow b \backslash (c \oslash a) & \text{iff} & (a \otimes c) \oslash b \leftarrow a \otimes (c \oslash b) & (J_3)'
\end{array}
$$

We work with the $(I_n)', (J_n)'$ forms. Writing $\mathcal{G}^{\uparrow}$ for the four J principles, and $\mathcal{G}^{\downarrow}$ for the converse I principles, we obtain the following landscape:

$$
\begin{tikzcd}
 & & \mathbf{LG}_{\emptyset} + \mathcal{G}^{\downarrow,\uparrow} & & \\
\text{(Class I)} & \mathbf{LG}_{\emptyset} + \mathcal{G}^{\downarrow} \arrow[ur] & & \mathbf{LG}_{\emptyset} + \mathcal{G}^{\uparrow} \arrow[ul] & \text{(Class IV)} \\
 & & \mathbf{LG}_{\emptyset} \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

25. The group of Grishin interactions

Initial presentations of Grishin's work (Lambek 93, Goré 98) publicized only half of the interaction principles ($G1$, $G3$). This breaks the $\bowtie$ symmetry.

What relates the four members of the J family?

$$
\begin{array}{lll@{\qquad}lr}
(G1) & (A \obslash B) \otimes C \rightarrow A \obslash (B \otimes C) & & C \otimes (B \oslash A) \rightarrow (C \otimes B) \oslash A & (G3) \\
(G2) & C \otimes (A \obslash B) \rightarrow A \obslash (C \otimes B) & & (B \oslash A) \otimes C \rightarrow (B \otimes C) \oslash A & (G4)
\end{array}
$$

$$
\begin{tikzcd}
G1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & G4 \arrow[d, leftrightarrow, "{(1,\bowtie)}"] \\
G2 \arrow[r, leftrightarrow, "{(\bowtie,1)}"'] & G3
\end{tikzcd}
$$

(p, q) is the transformation acting as p on $\obslash ?$ $(? \oslash)$ and as q on $? \otimes$ $(\otimes ?)$.

Along the diagonals, we have $(\bowtie, \bowtie) = (1, \bowtie)(\bowtie, 1) = (\bowtie, 1)(1, \bowtie)$.

26. The Klein group

Let us write $\sharp = (1, \bowtie)$, $\flat = (\bowtie, 1)$ and $\sim = (\bowtie, \bowtie)$. Together with the identity transformation $\sharp$, $\flat$ and $\sim$ constitute the Klein group. This is the smallest non-cyclic abelian group.

	1	$\sharp$	$\flat$	$\sim$
1	1	$\sharp$	$\flat$	$\sim$
$\sharp$	$\sharp$	1	$\sim$	$\flat$
$\flat$	$\flat$	$\sim$	1	$\sharp$
$\sim$	$\sim$	$\flat$	$\sharp$	1

$$
\begin{array}{llll}
(G1) & (A \oslash B) \otimes C \to A \oslash (B \otimes C) & C \otimes (B \oslash A) \to (C \otimes B) \oslash A & (G3) \\
(G2) & C \otimes (A \oslash B) \to A \oslash (C \otimes B) & (B \oslash A) \otimes C \to (B \otimes C) \oslash A & (G4)
\end{array}
$$

27. Structural semantics

We extend the completeness result for asymmetric Lambek calculus to **LG**.

Truth conditions Fission ($\oplus$) and its residuals:

$$
\begin{array}{lll}
x \Vdash A \oplus B & \text{iff} & \forall yz.R_{\oplus}xyz \text{ implies } (y \Vdash A \text{ or } z \Vdash B) \\
y \Vdash C \oslash B & \text{iff} & \exists xz.R_{\oplus}xyz \text{ and } z \nVdash B \text{ and } x \Vdash C \\
z \Vdash A \obslash C & \text{iff} & \exists xy.R_{\oplus}xyz \text{ and } y \nVdash A \text{ and } x \Vdash C
\end{array}
$$

Compare Fusion/composition ($\otimes$) and its residuals:

$$
\begin{array}{lll}
x \Vdash A \otimes B & \text{iff} & \exists yz.R_{\otimes}xyz \text{ and } y \Vdash A \text{ and } z \Vdash B \\
y \Vdash C \,/\, B & \text{iff} & \forall xz.(R_{\otimes}xyz \text{ and } z \Vdash B) \text{ implies } x \Vdash C \\
z \Vdash A \setminus C & \text{iff} & \forall xy.(R_{\otimes}xyz \text{ and } y \Vdash A) \text{ implies } x \Vdash C
\end{array}
$$

Remark In the minimal symmetric system $\mathbf{LG}_{\emptyset}$, fission $R_{\oplus}$ and merge $R_{\otimes}$ are distinct relations.

28. Henkin construction

To establish completeness, one uses a Henkin model with (weak) filters $\mathcal{F}_\uparrow$ as worlds: sets of formulas closed under derivability.

The set of filters $\mathcal{F}_\uparrow$ is closed under the following $\widehat{\cdot}$ operations, in terms of which one then defines the canonical model.

$$X \mathbin{\widehat{\otimes}} Y = \{C \mid \exists A, B \ (A \in X \text{ and } B \in Y \text{ and } A \otimes B \to C)\}$$

$$X \mathbin{\widehat{\oslash}} Y = \{B \mid \exists A, C \ (A \not\in X \text{ and } C \in Y \text{ and } A \oslash C \to B\}$$

To lift the type-forming operations to the corresponding operations in $\mathcal{F}_\uparrow$, let $\lfloor A \rfloor$ be the principal filter generated by A, i.e. $\lfloor A \rfloor = \{B \mid A \to B\}$ and $\lceil A \rceil$ its principal ideal, i.e. $\lceil A \rceil = \{B \mid B \to A\}$. Writing $X^\sim$ for the complement of X, we have

$$(\dagger) \quad \lfloor A \otimes B \rfloor = \lfloor A \rfloor \mathbin{\widehat{\otimes}} \lfloor B \rfloor \qquad\qquad (\ddagger) \quad \lfloor A \oslash C \rfloor = \lceil A \rceil^\sim \mathbin{\widehat{\oslash}} \lfloor C \rfloor$$

29. LG: completeness

Theorem (Kurtonina and MM, MOL'07) If $\models A \to B$, then $A \to B$ is provable in the miminal symmetric Lambek-Grishin calculus $\mathbf{LG}_\emptyset$

The canonical model is defined as $\mathcal{M}^c = \langle W^c, R^c_\otimes, R^c_\oplus, V^c \rangle$ with

$$W^c = \mathcal{F}_\uparrow$$

$$R^c_\otimes XYZ \text{ iff } Y \mathbin{\widehat{\otimes}} Z \subseteq X$$

$$R^c_\oplus XYZ \text{ iff } Y \mathbin{\widehat{\oslash}} X \subseteq Z \text{ (iff } X \subseteq Y \mathbin{\widehat{\oplus}} Z)$$

$$V^c(p) = \{X \in W^c \mid p \in X\}$$

The proof follows immediately from the usual Truth Lemma, showing that for any formula $A \in \mathcal{F}$ and filter $X \in \mathcal{F}_\uparrow$, $X \Vdash A$ iff $A \in X$.

For $\mathbf{LG}_\emptyset + \mathcal{G}^{\downarrow,\uparrow}$, one has extended completeness for interpretations respecting the frame constraints corresponding to these postulates.

30. Decision procedure

In the algebraic presentation, Transitivity is the rule to produce new proofs from existing ones. The transitivity rule (cut) is not appropriate for automated proof search. Below a presentation that yields a decision procedure.

Combinator presentation We consider arrows $f : A \longrightarrow B$, with $A, B \in \mathcal{F}$. For every formula A, we have the identity arrow $1_A : A \longrightarrow A$. The Transitivity rule is composition of arrows:

$$\frac{f : A \longrightarrow B \quad g : B \longrightarrow C}{g \circ f : A \longrightarrow C}$$

Transitivity is an admissible rule in an axiomatization in terms of

- Shift rules: the (dual) residuation principles
- Reduce rules: monotonicity principles
- Interaction rules: the Grishin axioms in rule form

31. Shifting: (dual) residuation principles

$$\frac{f : A \otimes B \to C}{\triangleleft f : B \to A \backslash C} \qquad \frac{f : C \to A \oplus B}{\blacktriangleleft f : A \oslash C \to B}$$

$$\frac{f : A \otimes B \to C}{\triangleright f : A \to C / B} \qquad \frac{f : C \to A \oplus B}{\blacktriangleright f : C \oslash B \to A}$$

Mnemonic $\triangleright$, $\triangleleft$ shift the right, left coordinate of $\otimes$ over the turnstile; similarly, $\blacktriangleright$, $\blacktriangleleft$ for the right and left coordinate of $\oplus$.

Reverse shifts Add primed combinators $\triangleleft'$, $\triangleright'$, $\blacktriangleleft'$, $\blacktriangleright'$ to shift back.

32. Reduce: monotonicity laws

$$\frac{f : A \rightarrow B \quad g : C \rightarrow D}{f \otimes g : A \otimes C \rightarrow B \otimes D} \qquad \frac{f : A \rightarrow B \quad g : C \rightarrow D}{f \oplus g : A \oplus C \rightarrow B \oplus D}$$

$$\frac{f : A \rightarrow B \quad g : C \rightarrow D}{f/g : A/D \rightarrow B/C} \qquad \frac{f : A \rightarrow B \quad g : C \rightarrow D}{f \oslash g : A \oslash D \rightarrow B \oslash C}$$

$$\frac{f : A \rightarrow B \quad g : C \rightarrow D}{g\backslash f : D\backslash A \rightarrow C\backslash B} \qquad \frac{f : A \rightarrow B \quad g : C \rightarrow D}{g \obslash f : D \obslash A \rightarrow C \obslash B}$$

Remark $\otimes$, $\oplus$ are isotonic in both coordinates; the slashes isotonic in the numerator, antitonic in the denominator. These properties are easily derived from transitivity and the (dual) residuation principles.

33. Grishin interaction

We also encapsule the Grishin axioms in derived inference rules.

$$
\frac{f : A \obslash (B \otimes C) \rightarrow D}{\urcorner f : (A \obslash B) \otimes C \rightarrow D}
\qquad
\frac{f : (A \otimes B) \oslash C \rightarrow D}{\ulcorner f : A \otimes (B \oslash C) \rightarrow D}
$$

$$
\frac{f : B \obslash (A \otimes C) \rightarrow D}{\lrcorner f : A \otimes (B \obslash C) \rightarrow D}
\qquad
\frac{f : (A \otimes C) \oslash B \rightarrow D}{\llcorner f : (A \oslash B) \otimes C \rightarrow D}
$$

Remark This decision procedure can be seen as a compact version of display logic (Goré 98): we drop the explicit rewriting of structural occurrences of connectives into sequent punctuation. Structural occurrences are determined by polarity ($\cdot^{\bullet}$ input, $\cdot^{\circ}$ output).

$$
\begin{array}{rcl}
(A \otimes B)^{\bullet} & = & A^{\bullet} \otimes B^{\bullet} \\
(C/B)^{\bullet} & = & C^{\bullet}/B^{\circ} \\
(A\backslash C)^{\bullet} & = & A^{\circ}\backslash C^{\bullet}
\end{array}
\qquad
\begin{array}{rcl}
(A \otimes B)^{\circ} & = & A^{\circ} \otimes B^{\circ} \\
(C/B)^{\circ} & = & C^{\circ} / B^{\bullet} \\
(A\backslash C)^{\circ} & = & A^{\bullet} \backslash C^{\circ}
\end{array}
$$

34. Symmetries: proofs

$$\infty \quad \frac{h/g \qquad f \otimes g \qquad f\backslash h}{\overline{g \oslash h \qquad g \oplus f \qquad h \oslash f}} \qquad\qquad \infty \quad \frac{\lhd f \qquad \rhd f \qquad \lhd' f \qquad \rhd' f}{\overline{\blacktriangleright f \qquad \blacktriangleleft f \qquad \blacktriangleright' f \qquad \blacktriangleleft' f}}$$

Example $(\blacktriangleleft' \rhd \urcorner \blacktriangleleft \rhd' 1_{(c \oplus b)/a})^{\infty} = \rhd' \blacktriangleleft \urcorner^{\infty} \rhd \blacktriangleleft' 1_{a \oslash (b \otimes c)}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(c \oplus b)/a \to (c \oplus b)/a}{((c \oplus b)/a) \otimes a \to c \oplus b}\,\rhd'}{c \oslash (((c \oplus b)/a) \otimes a) \to b}\,\blacktriangleleft}{(c \oslash ((c \oplus b)/a)) \otimes a \to b}\,\urcorner}{c \oslash ((c \oplus b)/a) \to b/a}\,\rhd}{(c \oplus b)/a \to c \oplus (b/a)}\,\blacktriangleleft' \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \oslash (b \otimes c) \to a \oslash (b \otimes c)}{b \otimes c \to a \oplus ((a \oslash (b \otimes c))}\,\blacktriangleleft'}{b \to (a \oplus ((a \oslash (b \otimes c)))/c}\,\rhd}{b \to a \oplus ((a \oslash (b \otimes c))/c)}\,\urcorner^{\infty}}{a \oslash b \to (a \oslash (b \otimes c))/c}\,\blacktriangleleft}{(a \oslash b) \otimes c \to a \oslash (b \otimes c)}\,\rhd'$$

35. Illustrations

We give some highly simplified examples suggesting how in the symmetric setting of **LG**, the expressive limitations of Lambek's original systems can be overcome.

Strategy For non-local binding, displacement, we start from lexical type assignments from which the original assignments are derivable:

someone	$(s \oslash s) \obslash np$	$\vdash \; s/(np\backslash s)$
which	$(n\backslash n)/((s \oslash s) \oplus (s/np))$	$\vdash \; (n\backslash n)/(s/np)$

36. GQ: local vs non-local scope construal

We leave the local construal as an exercise. Below the non-local construal.

$$\dfrac{\dfrac{np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s))) \vdash s \quad s \vdash (s \oslash s) \oplus s}{np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s))) \vdash (s \oslash s) \oplus s}}{(s \oslash s) \obslash (np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s)))) \vdash s}$$

$$\dfrac{\vdots}{\underbrace{np}_{\text{Alice}} \otimes (\underbrace{((np\backslash s)/s)}_{\text{thinks}} \otimes (\underbrace{((s \oslash s) \obslash np)}_{\text{someone}} \otimes \underbrace{(np\backslash s)}_{\text{left}})) \vdash s}$$

- The $(s \oslash s)$ moves upwards through $\otimes$ structure, leaving np behind
- When $(s \oslash s)$ has reached the top, it can jump to the rhs by means of the dual residuation principle.

37. Displacement

Example This time, we use the Grishin principles plus converses to derive '(movie which) John (np) saw ($(np\backslash s)/np = tv$) on TV ($(np\backslash s)\backslash(np\backslash s) = adv$)'.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{np \otimes ((tv \otimes np) \otimes adv) \vdash s \quad s \vdash (s \oslash s) \oplus s}{np \otimes ((tv \otimes np) \otimes adv) \vdash (s \oslash s) \oplus s}\ \text{trans}}{tv \vdash ((np\backslash(s \oslash s))/adv) \oplus (s/np)}\ Gn,\text{rp}}{np \otimes ((tv \oslash (s/np)) \otimes adv) \vdash s \oslash s}\ \text{rp,drp}}{(np \otimes (tv \otimes adv)) \oslash (s/np) \vdash s \oslash s}\ Gn^{-1}}{np \otimes (tv \otimes adv) \vdash (s \oslash s) \oplus (s/np)}\ \text{drp}
$$

Compare structural rules under modal control.

$$
\frac{\Gamma[\Delta \circ B] \Rightarrow C}{\Gamma[\Delta] \Rightarrow (C \oslash C) \oplus (C/B)}\ \dagger \qquad \frac{\Gamma[\Delta \circ B] \Rightarrow C}{\Gamma[\Delta] \Rightarrow C/\Diamond\Box B}\ \ddagger
$$

38. What have you learned today?

- Duals for the Lambek connectives: $\oslash, \oplus, \obslash$
- $\mathbf{LG}_{\emptyset}$: the pure residuation logic for the extended language
- Structure preserving interaction between the $\otimes$ and $\oplus$ families
- Completeness w.r.t. relational semantics
- Decision procedure for **LG** theoremhood

39. More To Explore

In the course materials you find

- Cut elimination for **LG**: Ch 2, Appendix
- Relational completeness: Ch 4
- For the intrepid: Ch 1, Section 2, is the central section for the landscape of Lambek-Grishin systems.

The References of Ch 2 contain useful references for further reading (both representing only half of the interaction principles):

- Lambek (1993). First presentation of Grishin (1983) to a wider public.
- Goré (1998). Comprehensive display presentation of substructural logics.

40. Code

On `symcg.pbwiki.com` you will find a number of implementations that can assist you in gaining an understanding of **LG**. You may have to adjust the shell calls that do the typesetting of the output.

- A Java implementation by Gianluca Giorgolo. Both the top-down method discussed in class, and a bottom-up version based on Capelletti (2007).
- Prolog implementation (by MM) of the decision procedure discussed in class. Returns the shortest derivation for each available axiom linking.

Part III. Exercises

A.

A.1.

Using the preorder axioms and the dual residuation principles (Slide 18), prove the following.

$$(A \oplus B) \oslash B \rightarrow A \rightarrow (A \oslash B) \oplus B$$

A.2.

Derive the monotonicity rules for $\oslash, \oplus, \obslash$ from the preorder axioms and the dual residuation principles (Slide 18). In other words, given $A \rightarrow B$ and $C \rightarrow D$, show that

$$A \oslash D \rightarrow B \oslash C, \qquad A \oplus C \rightarrow B \oplus D, \qquad D \obslash A \rightarrow C \obslash B$$

B.

B.1.

On Slide 33, you find an abbreviated derivation for non-local scope construal (repeated here) using Transitivity (cut). Give a cut-free derivation, using the decision procedure of Slide 29ff.

$$\dfrac{\dfrac{\dfrac{np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s))) \vdash s \quad s \vdash (s \oslash s) \oplus s}{np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s))) \vdash (s \oslash s) \oplus s}}{(s \oslash s) \obslash (np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s)))) \vdash s}}{np \otimes \; (((np\backslash s)/s) \otimes (((s \oslash s) \obslash np) \otimes (np\backslash s))) \vdash s}$$

B.2.

Do the same for the local scope construal.

C.

C.1.

Below six equivalent forms of interaction principle $(J_3)'$ of Slide 23, from Grishin's Class IV (section 2.7, and 2.5 for the recipe; our notation). Assuming $(J_3)'$, prove the other five.

$$
\begin{array}{llll}
(1) & (b\backslash c) \oslash a \leq b\backslash(c \oslash a) & (4) & (a\backslash c) \obslash b \leq c \obslash (a \otimes b) \\
(2) & b\backslash(c \oplus a) \leq (b\backslash c) \oplus a & (5) & (a \oplus b)/c \leq a/(c \oslash b) \\
(3) & a \otimes (c \oslash b) \leq (a \otimes c) \oslash b & (6) & a \oslash (b \obslash c) \leq (c/a)\backslash b
\end{array}
$$

C.2.

Using the recipe of Grishin Section 2.5, give the six interderivable forms for cell (J_2) of Slide 22, with $\mu = \obslash ?$ and $\lambda = \backslash ?$.

D.

In CCG, one finds the following forward and backward forms of function composition (our notation), or crossed variants, used in the analysis of crossed dependencies. In the structure-preserving setting of **LG**, these are invalid: they destroy phrase structure and/or word order.

$$A/B \to (A/C)/(B/C) \qquad B\backslash A \to (C\backslash B)\backslash(C\backslash A)$$
$$A/B \to (C\backslash A)/(C\backslash B) \qquad B\backslash A \to (B/C)\backslash(A/C)$$

Show that the mixed forms of composition below (both plain and crossed) are in fact valid in **LG**. In Thursday's class, you will see how they can be used in the analysis of Dutch Verb Raising.

$$A/B \to (A \oslash C)/(B \oslash C) \qquad B\backslash A \to (C \obslash B)\backslash(C \obslash A)$$
$$A/B \to (C \obslash A)/(C \obslash B) \qquad B\backslash A \to (B \oslash C)\backslash(A \oslash C)$$

E.

In the Appendix of Ch 2 of the course notes, you find the cut elimination algorithm for **LG**. Induction is on the complexity (number of connectives) in a cut inference: one has to show that any uppermost cut (i.e. cut application which doesn't itself involve applications of cut) can be replaced by one or more cuts of lower complexity.

Fig 2.5 gives the transformation for a Grishin inference in the right cut premise.

$$\dfrac{E \longrightarrow (A \obslash B) \otimes C \qquad \dfrac{A \obslash (B \otimes C) \longrightarrow D}{(A \obslash B) \otimes C \longrightarrow D}}{E \longrightarrow D}\ \text{Trans}$$

How do you deal with the case below?

$$\dfrac{\dfrac{A \obslash (B \otimes C) \longrightarrow D}{(A \obslash B) \otimes C \longrightarrow D} \qquad D \longrightarrow E}{(A \obslash B) \otimes C \longrightarrow E}\ \text{Trans}$$